A Penetrable Dielectric Waveguide with Periodically Varying Circular Cross Section

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Abstract —For a penetrable dielectric waveguide with a periodically varying circular cross section, the modes that are exponentially decreasing in the cladding are considered. Their axial wavenumbers are determined by the null field approach, and some plots are given showing their frequency dependence. From the numerical results, it is observed that two modes propagating in opposite directions interact destructively when the real parts of their axial wavenumbers differ by a multiple of the wavenumber of the corrugations. Both an upper and a lower cutoff frequency exists above (below) which only leaky modes exist.

I. Introduction

In the present paper, we study a penetrable dielectric cylinder with a periodically varying circular cross section. This waveguide, with interior dielectric constant ϵ_- and permeability μ_-, is embedded in an infinite medium with the corresponding exterior constants ϵ_+ and μ_+, where $\mu_+\epsilon_+ < \mu_-\epsilon_-$. The geometry is depicted in Fig. 1, where the wall is denoted by S and the outward unit normal by $\hat{n}$.

We are primarily interested in guided waves, i.e., modes that are exponentially decreasing outside S. These modes can be of two types, either propagating or exponentially decaying along the waveguide. The existence of frequency domains in which no propagating modes exist is used in practical devices such as the corrugated feedback laser [1]. For further practical applications and a more general treatment of the two-dimensional corrugated waveguide, we refer to Elachi and Yeh [2] and Peng *et al.* [3]. In addition to these two kinds of modes which are bounded to the waveguide, there also exist leaky modes, which are exponentially increasing outside S and have only a restricted physical interpretation.

The straight dielectric waveguide has been addressed by an extensive number of authors from both a theoretical and an experimental point of view; we refer to Clarricoats [4] and Marcuse [5] and the references cited therein. There exist some recent investigations that consider dielectric waveguides with a periodically varying circular cross section [6], [7].

In the present paper, we make use of the null-field approach (T-matrix method). This method was introduced by Waterman [8], [9] for electromagnetic scattering problems and has also been extended to the scattering by a periodic surface by the same author [10]. This method has subsequently been used to investigate several different problems involving corrugated surfaces [11]–[15]. In the null-field approach, it is also possible to consider a scatterer inside or outside the waveguide [16], [17] or a finite cladding.

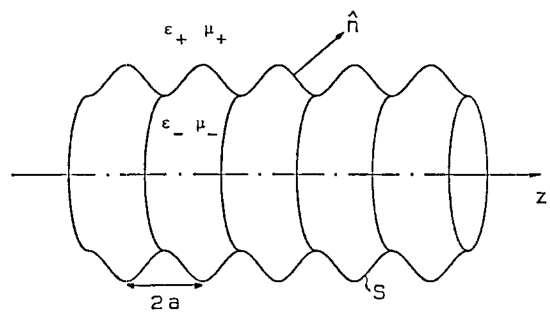

Fig. 1. The geometry of the corrugated waveguide with the period of the corrugations equal to the mean radius.

From the numerical results shown below, it is observed that when the axial wavenumber exceeds the wavenumber of the wall, all modes become leaky. Thus, the corrugation introduces an upper cutoff frequency, above which no exponentially decreasing (in the radial direction) solution exists in the cladding. Furthermore, two types of stopbands exist, where the solution is exponentially decreasing along the cylinder. The first one appears when the real parts of two axial wavenumbers differ by a multiple of the wavelength of the wall. The emergence of this type of stopband is just an effect of the ordinary Bragg coupling between modes propagating in opposite directions. In the second type of stopband, the axial wavenumbers do not satisfy any simple relation.

II. Method of Solution

Consider a cylindrical surface S with periodically varying circular cross section and introduce cylindrical coordinates (ρ, ϕ, z). The surface is assumed penetrable, and all quantities inside (outside) the cylinder will be indicated by a minus (plus) sign. The time harmonic dependence $e^{-i\omega t}$ is assumed and suppressed throughout this paper. The electric fields $\vec{E}_\pm(\vec{r})$ in a homogeneous, isotropic, lossless

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medium satisfy the homogeneous vector Helmholtz equation

$$\nabla\times\nabla\times\vec{E}_\pm(\vec{r})-k_\pm^2\vec{E}_\pm(\vec{r})=0 \qquad (1)$$

where $k_\pm^2=\omega^2\epsilon_\pm\mu_\pm$; $\epsilon_\pm$ are the dielectric constants and $\mu_\pm$ the permeabilities. The boundary conditions at S require that the tangential parts of the electric and magnetic fields be continuous

$$\hat{n}\times\vec{E}_-(\vec{r})=\hat{n}\times\vec{E}_+(\vec{r})$$
$$\hat{n}\times\left(\nabla\times\vec{E}_-(\vec{r})\right)=\gamma\hat{n}\times\left(\nabla\times\vec{E}_+(\vec{r})\right) \qquad (2)$$

where $\gamma=\mu_-/\mu_+$, and $\hat{n}$ is the normal unit vector pointing from the boundary into the exterior domain. The magnetic fields satisfy the same wave equation but with $\gamma=\epsilon_-/\epsilon_+$ in the boundary condition. Since they are formally equivalent, it is sufficient to only consider electric fields explicitly.

The introduction of a unit source, or any superposition of such sources, outside S and an application of Green's theorem in the two regions give

$$\vec{E}_+^i(\vec{r})+\nabla\times\int_S G(\vec{r},\vec{r}';k_+)\hat{n}'\times\vec{E}_+(\vec{r}')\,ds' + k_+^{-2}\nabla\times\nabla\times\int_S G(\vec{r},\vec{r}';k_+)\,\hat{n}'\times\left[\nabla'\times\vec{E}_+(\vec{r}')\right]ds'=\begin{cases}\vec{E}_+(\vec{r}) & \vec{r}\text{ outside } S\\ 0 & \vec{r}\text{ inside } S\end{cases} \qquad (3a)$$

$$-\nabla\times\int_S G(\vec{r},\vec{r}';k_-)\hat{n}'\times\vec{E}_-(\vec{r}')\,ds' - k_-^{-2}\nabla\times\nabla\times\int_S G(\vec{r},\vec{r}';k_-)\,\hat{n}'\times\left[\nabla'\times\vec{E}_-(\vec{r}')\right]ds'=\begin{cases}\vec{E}_-(\vec{r}) & \vec{r}\text{ inside } S\\ 0 & \vec{r}\text{ outside } S\end{cases} \qquad (3b)$$

where $\vec{E}_+^i(\vec{r})$ is the incoming field from the sources and $G(\vec{r},\vec{r}';k)$ is the free-space Green function:

$$G(\vec{r},\vec{r}';k)=\frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}. \qquad (4)$$

Now introduce the cylindrical vector wave functions $\vec{\chi}_{\tau\sigma m}$, defined as

$$\vec{\chi}_{\tau\sigma m}(h,k;\vec{r}) =\sqrt{\frac{\epsilon_m}{8\pi}}\,\frac{k}{q}(k^{-1}\nabla\times)^\tau\left[\hat{z}H_m^{(1)}(q\rho)\begin{bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{bmatrix}e^{ihz}\right] \qquad (5)$$

where $\epsilon_m=2-\delta_{m0}$; $H_m^{(1)}$ denotes the Hankel function of the first kind and order m; $q=(k^2-h^2)^{1/2}$ with $\operatorname{Im}q\geqslant 0$; $\tau=1,2$ (for TE and TM modes); and $\sigma=e,o$ [for $\cos(m\phi)$ and $\sin(m\phi)$]. For convenience, we will henceforth employ the multi-index $j=\tau\sigma m$. The expansion of the free-space Green dyadic is [16]

$$\overleftrightarrow{I}G(\vec{r},\vec{r}';k) =i\sum_j\int_{-\infty}^{\infty}\operatorname{Re}\vec{\chi}_j^\dagger(h,k;\vec{r}_<)\vec{\chi}_j(h,k;\vec{r}_>)\,dh+\overleftrightarrow{I}_{\rm irr} \qquad (6)$$

where $\overleftrightarrow{I}_{\rm irr}$ is an irrotational dyadic; $\overleftrightarrow{I}$ is the unit dyadic; $\vec{r}_<$ ($\vec{r}_>$) denotes the $\vec{r},\vec{r}'$ with the smaller (greater) value of ρ and ρ'; and $\operatorname{Re}\vec{\chi}_j$ is obtained from $\vec{\chi}_j$ by replacing the Hankel function by a Bessel function in (5). The dagger indicates a shift from e^{ihz} to e^{-ihz} before the curl operators in (5) are applied. This dagger could equally well be situated on $\vec{\chi}_j$ as on $\operatorname{Re}\vec{\chi}_j$.

We expand the incoming field in $\{\operatorname{Re}\vec{\chi}_j\}$ inside the inscribed cylinder of S

$$\vec{E}_+^i(\vec{r})=\sum_j\int_{-\infty}^{\infty}a_j(h)\operatorname{Re}\vec{\chi}_j(h,k_+;\vec{r})\frac{dh}{k_+} \qquad (7)$$

and the scattered field in $\{\vec{\chi}_j\}$ outside the circumscribed cylinder of S

$$\vec{E}_+^s(\vec{r})=\vec{E}_+(\vec{r})-\vec{E}_+^i(\vec{r})=\sum_j\int_{-\infty}^{\infty}f_j(h)\vec{\chi}_j(h,k_+;\vec{r})\frac{dh}{k_+} \qquad (8)$$

where the poles of $f_j(h)$ will eventually determine the axial wavenumbers of the guided waves. We insert the expansion of the incoming field into the lower equation of (3a), the expansion of the scattered field into the upper equation of (3a), the boundary conditions into the lower equation of (3b), and the expansion of the Green function into both of the integral representations. The resulting relations are

$$\sum_j\int_{-\infty}^{\infty}a_j(h)\operatorname{Re}\vec{\chi}_j(h,k_+;\vec{r})\frac{dh}{k_+} =-\nabla\times\int_S i\sum_j\int_{-\infty}^{\infty}\operatorname{Re}\vec{\chi}_j(h,k_+;\vec{r})\vec{\chi}_j^\dagger(h,k_+;\vec{r}')\cdot\left[\hat{n}'\times\vec{E}_+(\vec{r}')\right]dh\,ds' - k_+^{-2}\nabla\times\nabla \times\int_S i\sum_j\int_{-\infty}^{\infty}\operatorname{Re}\vec{\chi}_j(h,k_+;\vec{r})\vec{\chi}_j^\dagger(h,k_+;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{E}_+(\vec{r}')\right)'\right]dh\,ds' \qquad \vec{r}\text{ inside } S$$

$$\sum_j\int_{-\infty}^{\infty}f_j(h)\vec{\chi}_j(h,k_+;\vec{r})\frac{dh}{k_+} =\nabla\times\int_S i\sum_j\int_{-\infty}^{\infty}\vec{\chi}_j(h,k_+;\vec{r})\operatorname{Re}\vec{\chi}_j^\dagger(h,k_+;\vec{r}')\cdot\left[\hat{n}'\times\vec{E}_+(\vec{r}')\right]dh\,ds' + k_+^{-2}\nabla\times\nabla \times\int_S i\sum_j\int_{-\infty}^{\infty}\vec{\chi}_j(h,k_+;\vec{r})\operatorname{Re}\vec{\chi}_j^\dagger(h,k_+;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{E}_+(\vec{r}')\right)\right]dh\,ds' \qquad \vec{r}\text{ outside } S$$

$$-\nabla\times\int_S\sum_j\int_{-\infty}^{\infty}\vec{\chi}_j(h,k_-;\vec{r})\operatorname{Re}\vec{\chi}_j^\dagger(h,k_-;\vec{r}')\cdot\left[\hat{n}'\times\vec{E}_+(\vec{r}')\right]dh\,ds' =k_-^{-2}\gamma\nabla\times\nabla\times\int_S\sum_j\int_{-\infty}^{\infty}\vec{\chi}_j(h,k_-;\vec{r})\operatorname{Re}\vec{\chi}_j^\dagger(h,k_-;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{E}_+(\vec{r}')\right)\right]dh\,ds' \qquad \vec{r}\text{ outside } S. \qquad (9)$$

From the definition of the cylindrical vector wave functions, it follows that

$$\nabla \times \vec{\chi}_{\tau\sigma m}(h,k;\vec{r}) = k\vec{\chi}_{\tau'\sigma m}(h,k;\vec{r}) \quad (10)$$

with $\tau' \neq \tau$. The linear independence of the $\{\vec{\chi}_j\}$ and $\{\mathrm{Re}\,\vec{\chi}_j\}$ systems then implies

$$a_j(h) = -ik_+\int_S \left[\nabla' \times \vec{\chi}_j^\dagger(h,k_+;\vec{r}')\right]\cdot\left[\hat{n}'\times\vec{E}_+(\vec{r}')\right] ds' - ik_+\int_S \vec{\chi}_j^\dagger(h,k_+;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{E}_+(\vec{r}')\right)\right] ds'$$

$$f_j(h) = ik_+\int_S \left[\nabla' \times \mathrm{Re}\,\vec{\chi}_j^\dagger(h,k_+;\vec{r}')\right]\cdot\left[\hat{n}'\times\vec{E}_+(\vec{r}')\right] ds' + ik_+\int_S \mathrm{Re}\,\vec{\chi}_j^\dagger(h,k_+;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{E}_+(\vec{r}')\right)\right] ds'$$

$$\int_S \left[\nabla' \times \mathrm{Re}\,\vec{\chi}_j^\dagger(h,k_-;\vec{r}')\right]\cdot\left[\hat{n}'\times\vec{E}_+(\vec{r}')\right] ds' = -\gamma\int_S \mathrm{Re}\,\vec{\chi}_j^\dagger(h,k_-;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{E}_+(\vec{r}')\right)\right] ds'. \quad (11)$$

We now expand the surface fields appearing in these equations in some complete systems

$$\hat{n}'\times\left[\nabla'\times\vec{E}_+(\vec{r}')\right] = \sum_{j'}\int_{-\infty}^{\infty}\alpha_{j'}(h')\,\hat{n}'\times\left[\nabla'\times\vec{\zeta}_{j'}^{\alpha}(h';\vec{r}')\right]\frac{dh'}{k_+}$$

$$\hat{n}'\times\vec{E}_+(\vec{r}') = \sum_{j'}\int_{-\infty}^{\infty}\beta_{j'}(h')\,\hat{n}'\times\vec{\zeta}_{j'}^{\beta}(h';\vec{r}')\frac{dh'}{k_+}. \quad (12)$$

The systems $\{\vec{\zeta}_j^{\alpha}\}$ and $\{\vec{\zeta}_j^{\beta}\}$ can, for instance, be selected as the trigonometric system

$$\vec{\zeta}_j = \begin{cases} (\epsilon_m/8\pi)^{1/2}\begin{bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{bmatrix} e^{ihz}\phi & \tau = 1\\ (\epsilon_m/8\pi)^{1/2}\begin{bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{bmatrix} e^{ihz} n\times\phi & \tau = 2\end{cases} \quad (13)$$

the regular system

$$\vec{\zeta}_j = \mathrm{Re}\,\vec{\chi}_j(k_-) \quad (14a)$$

or the outgoing system

$$\vec{\zeta}_j = \vec{\chi}_j(k_+). \quad (14b)$$

The expansion (14a) gives a valid representation of the field in the whole interior domain and (14b) is valid in the whole exterior domain (cf. Millar [18]).

By inserting the expanded fields in (11), the coefficients of the incoming and scattered fields become

$$a_j(h) = i\sum_{j'}\int_{-\infty}^{\infty}\left[Q_{jj'}^{\beta+}(h,h')\beta_{j'}(h') + Q_{jj'}^{\alpha+}(h,h')\alpha_{j'}(h')\right]\frac{dh'}{k_+}$$

$$f_j(h) = -i\sum_{j'}\int_{-\infty}^{\infty}\left[\mathrm{Re}\,Q_{jj'}^{\beta+}(h,h')\beta_{j'}(h') + \mathrm{Re}\,Q_{jj'}^{\alpha+}(h,h')\alpha_{j'}(h')\right]\frac{dh'}{k_+}$$

$$0 = \sum_{j'}\int_{-\infty}^{\infty}\left[\mathrm{Re}\,Q_{jj'}^{\beta-}(h,h')\beta_{j'}(h') + \gamma\,\mathrm{Re}\,Q_{jj'}^{\alpha-}(h,h')\alpha_{j'}(h')\right]\frac{dh'}{k_+} \quad (15)$$

where

$$Q_{jj'}^{\alpha\pm}(h,h') = -k_+\int_S \vec{\chi}_j^\dagger(h,k_\pm;\vec{r}')\cdot\left[\hat{n}'\times\left(\nabla'\times\vec{\zeta}_{j'}^{\alpha}(h';\vec{r}')\right)\right] ds'$$

$$Q_{jj'}^{\beta\pm}(h,h') = -k_+^2\int_S \left[\nabla'\times\vec{\chi}_j^\dagger(h,k_\pm;\vec{r}')\right]\cdot\left[\hat{n}'\times\vec{\zeta}_{j'}^{\beta}(h';\vec{r}')\right] ds' \quad (16)$$

and in $\mathrm{Re}\,Q_{jj'}$, $\mathrm{Re}\,\vec{\chi}_j^\dagger$ replaces $\vec{\chi}_j^\dagger$. Eliminating the $\alpha-$ and $\beta-$ coefficients in (15) gives the scattered field in terms of the incoming field, and this relation can be written as (in obvious operator notation)

$$\vec{f} = \left[\gamma\,\mathrm{Re}\,Q^{\beta+}(\mathrm{Re}\,Q^{\beta-})^{-1}\mathrm{Re}\,Q^{\alpha-} - \mathrm{Re}\,Q^{\alpha+}\right]\cdot\left[-\gamma Q^{\beta+}(\mathrm{Re}\,Q^{\beta-})^{-1}\mathrm{Re}\,Q^{\alpha-} + Q^{\alpha+}\right]^{-1}\vec{a}. \quad (17)$$

Thus, the scattering problem is formally solved. The poles of $f_j(h)$ determine the axial wavenumbers h of the guided waves [11] and will occur where

$$\det\left(\gamma Q^{\beta+}(\mathrm{Re}\,Q^{\beta-})^{-1}\mathrm{Re}\,Q^{\alpha-} - Q^{\alpha+}\right)(h) = 0. \quad (18)$$

The computation of the Q matrix is discussed by Boström [11], so only the final results are given here. The rotational symmetry of this specific problem gives a decoupling between even ($\tau\sigma = 1\mathrm{o}, 2\mathrm{e}$) and odd ($\tau\sigma = 1\mathrm{e}, 2\mathrm{o}$) modes and a Kronecker delta $\delta_{mm'}$ when the azimuthal integration is performed. The periodicity in the z direction makes it possible to reduce the integration to one period $(-a, a)$, and after a few substitutions, the Q matrices reappear as

$$Q_{\tau n,\tau' n'}^{(m)\alpha\pm}(h) = -k_+\pi/a\int_{-a}^{a}\vec{\chi}_{\tau m}^\dagger(h+n\pi/a,k_\pm;z)\cdot\left[\hat{n}\times\left(\nabla\times\vec{\zeta}_{\tau' m}^{\alpha}(h+n'\pi/a;z)\right)\right]\frac{dz}{n_\rho}$$

$$Q_{\tau n,\tau' n'}^{(m)\beta\pm}(h) = -k_+\pi/a\int_{-a}^{a}\left[\nabla\times\vec{\chi}_{\tau m}^\dagger(h+n\pi/a,k_\pm;z)\right]\cdot\left[\hat{n}\times\vec{\zeta}_{\tau' m}^{\beta\pm}(h+n'\pi/a;z)\right]\frac{dz}{n_\rho} \quad (19)$$

where $n_\rho = \hat{n}\cdot\hat{\rho}$; the ϕ dependence of $\vec{\zeta}_j$ and $\vec{\chi}_j$ is to be omitted; the azimuthal parity is suppressed; and n and n' run over all integers.

The axial wavenumber h of the guided waves is not unique, since if h gives a solution, then $h + n\pi/a$ (for any n) will also give a solution. For small corrugations, Marcuse and Derosier [19] have proved that modes become exponentially decreasing along the cylinder when their axial wavenumbers differ by a multiple of the wavelength of the wall ($n\pi/a$) and that they propagate in opposite directions.

The essential part of the exterior wave can be written as a summation of terms of the form

$$H_m^{(1)}(q_n \rho) e^{in\pi z/a} \qquad \text{for } n = 0, 1, -1, 2, -2, \cdots \tag{20}$$

where the arguments in the Hankel functions are $q_n^2 = k_+^2 - (h + n\pi/a)^2$. In order to have an exponentially decreasing mode outside the cylinder, it is sufficient that $\operatorname{Im} q_n > 0$ for all n. If the imaginary part of h is zero, $(h + n\pi/a)^2$ must be greater than k_+^2 for all n, which is only possible for k_+ less than $\pi/2a$.

III. Numerical Results

For the numerical illustrations, we have taken the wall corrugation to vary sinusoidally, with the period of the corrugations equal to the mean radius:

$$\rho(z) = a + d\cos(2\pi z/a). \tag{21}$$

Further, we impose the commonly valid restriction $\mu_- = \mu_+$ and only investigate the axially symmetric mode of lowest order. Four different corrugations are considered: $d/a =$ 0.00, 0.10, 0.20, and 0.30; in the figures, they are denoted by 0, 1, 2, and 3, respectively. For the plotted solutions, all $q_n = (k_+^2 - (h + n\pi/a)^2)^{1/2}$ have $\operatorname{Im} q_n > 0$, which implies that the solutions are exponentially decreasing outside S. For a computation of leaky modes with $\operatorname{Im} q_n < 0$ for some n, we refer to Lundqvist [12], where the exterior acoustic problem is investigated.

In Fig. 2, the real part of the axial wavenumber is plotted as a function of the normalized interior frequency $k_- a$ for the contrast $\epsilon_+/\epsilon_- = 0.6$, and in Fig. 3 the corresponding imaginary part is presented. The mode is cut on when $k_+^2 = h^2$, with $k_+^2 = k_-^2 \epsilon_+/\epsilon_-$ (which is the straight dotted line), and this occurs at $k_- a = 3.80$, 3.78, 3.70, and 3.66 for corrugations $d/a = 0.00$, 0.10, 0.20, and 0.30, respectively. Below cut-on, only the solution in the straight waveguide is exponentially decreasing outside the wall. The modes in the corrugated waveguides become leaky and have not been further investigated. Above cut-on, the modes are propagating for only a short frequency range before the axial wavenumbers become complex, and this occurs when the normalized axial wavenumber h/k_- equals $\pi/k_- a$. In the stopband around $k_- a = 4.0$, the axial wavenumber h is constant and equals π/a (note that we have plotted h/k_- and not just h), which is half the wavenumber of the corrugations. This stopband corresponds to the ordinary Bragg reflection, which couples modes propagating in opposite directions. For increasing

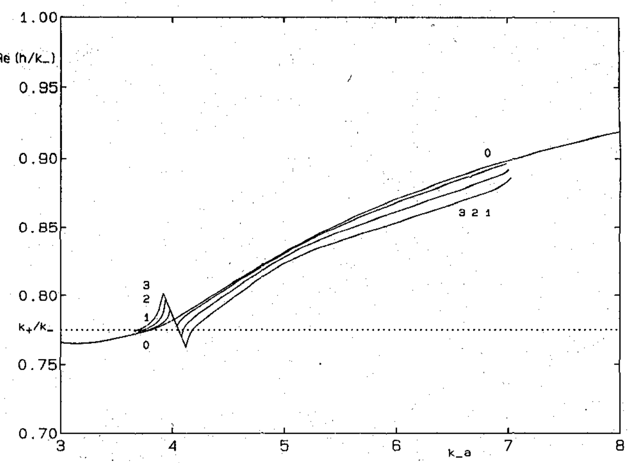

Fig. 2. The real part of the axial wavenumber as a function of the interior frequency for corrugations $d/a = 0.00$, 0.10, 0.20, and 0.30 with $\epsilon_+/\epsilon_- = 0.6$.

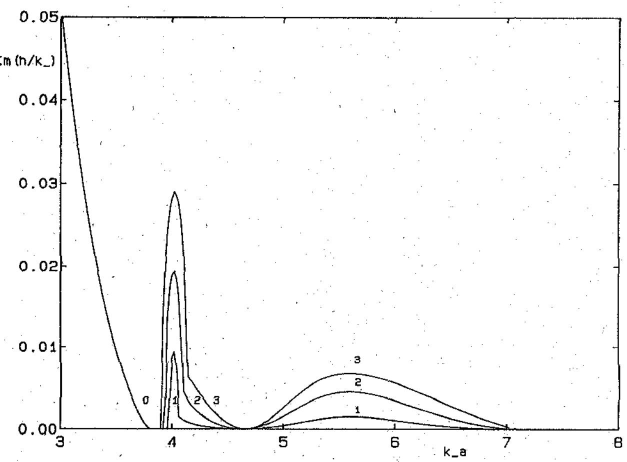

Fig. 3. The same as Fig. 2 but for the imaginary part.

corrugations, the imaginary part of h increases in this stopband, and the stopband also becomes broader. For axial wavenumbers greater than $\pi/k_- a$, the solution becomes exponentially decreasing along the waveguide. In Figs. 2 and 3, it is observed that the imaginary part does not vanish in the region $\pi/k_- a < h/k_- < 2\pi/k_- a$ (that is, from $k_- a \approx 4.0$ to ≈ 7.0), even though it becomes rather small just above the Bragg stopband.

When the axial wavenumber approaches $2\pi/a$, that is, $h/k_- \to 2\pi/k_- a$, the solution becomes exponentially increasing outside S; i.e., at least one q_n satisfies $\operatorname{Im} q_n < 0$ and disappears from the plotted sheet. Accordingly, for axial wavenumbers above $2\pi/a$, which happens around $k_- a = 7.0$, the solution is a leaky mode. The mode in the straight waveguide does not experience any such cutoff, of course.

In Fig. 4, the real part of the axial wavenumber is presented, and in Fig. 5 the imaginary part is given for the same corrugations as in Figs. 2 and 3 but in this case for the contrast $\epsilon_+/\epsilon_- = 0.1$. For this higher contrast, the cut-on frequencies seem to be quite independent of the corrugations considered and have decreased as compared to the previous contrast. The location of the Bragg stop-

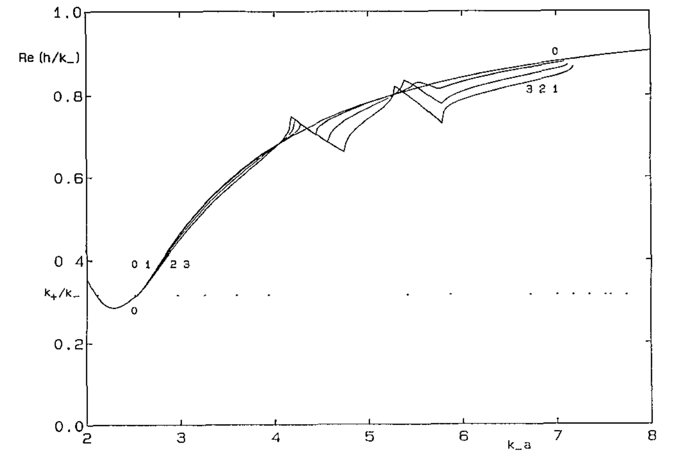

Fig. 4. The real part of the axial wavenumber as a function of the interior frequency for corrugations $d/a = 0.00$, 0.10, 0.20, and 0.30 with $\epsilon_+/\epsilon_- = 0.1$.

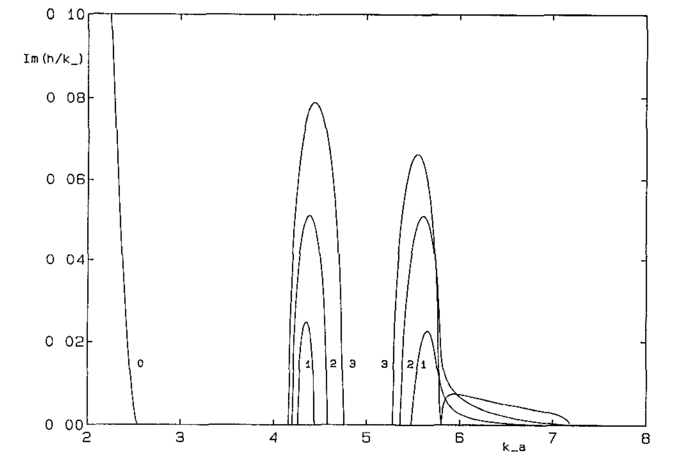

Fig. 5. The same as Fig. 4 but for the imaginary part.

band has moved upwards and has broadened, and the maximum imaginary parts have increased approximately three times. (Note the different scales in Fig. 3 and Fig. 5.)

In contrast to the previous case, a frequency domain, in which the solution is propagating, appears above the first stopband. As mentioned above, the sufficient condition to have an exponentially decreasing solution outside S is that $k_+ < |h + n\pi/a|$ for all integers n. For corrugation $d/a = 0.10$, the solution becomes exponentially decreasing along the waveguide ($\operatorname{Im} h = 0$) at about the point $k_+ = |h - \pi/a|$, and this occurs at $k_- a = 5.47$. But for the higher corrugations, the imaginary parts differ from zero before the sufficient condition above is violated, and for corrugation $d/a = 0.30$ the mode even becomes propagating after this second stopband before the sufficient condition is violated. In Fig. 5, it can be observed that the real part has a smoother appearance for the stopband limits associated with the sufficient condition above than for the other stopband limits. The beginning of the stopband for $d/a = 0.20$ at $k_- a = 5.37$, and the entire stopband around $k_- a = 5.6$ for $d/a = 0.30$ are a result of a Bragg reflection, and the stopband limits have the same edgy character as the first stopband.

The only difference between the two types of Bragg stopband is that the stopband around $k_- a = 4.2$ is a result of an interaction between the lowest mode with itself propagating in the opposite direction, but the second stopband around $k_- a = 5.6$ is a result of an interaction of the lowest mode with another mode. For a more careful investigation of these two kinds of stopbands, we refer the reader to Lundqvist and Boström [13].

In conclusion, we have demonstrated the usefulness of the null-field approach for investigating corrugated dielectric waveguides. This method also has the possibility of introducing a finite cladding or other generalizations. Three kinds of solutions are present in the numerical results. Solutions which are exponentially decreasing outside the waveguide can either be propagating or exponentially decreasing along the cylinder. The frequency regions where the mode is nonpropagating are of two types: one due to the ordinary Bragg-reflection condition and the other to the condition that the mode be exponentially decreasing in the cladding. Apart from these two kinds of solutions, leaky modes exist and they are the only solutions found for wavenumbers exceeding the wavenumbers of the wall corrugation or below cut-on.

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